306 J&K Lab 10 April 18, 2025 NAME SOLUTION

SERIES SOLUTIONS

1. Construction of the Taylor series of a known function

(i)

For the function

$f(x) = \frac{1}{2+x}$

calculate f', f'', f''', at general x and also at $x = 0$.

$f(0) = \frac{1}{2}$

$f'(x) = -(2+x)^{-2}$

$f'(0) = -\frac{1}{2^2} = -\frac{1}{4}$

$f''(x) = +2(2+x)^{-3}$

$f''(0) = \frac{2}{2^3} = \frac{1}{4}$

$f'''(x) = -6(2+x)^{-4}$

$f'''(0) = \frac{-6}{2^4} = -\frac{6}{16} = \frac{-3}{8}$

(ii)

Use the data you generated in part (i) to write down the degree-3 Taylor polynomial of f at 0.

$$f(x) \approx T_{f,0,3}(x) = \frac{1}{2} + \left(\frac{-1}{4}\right)x + \frac{1}{4}\frac{x^2}{2!} + \left(\frac{-3}{8}\right)\frac{x^3}{3!}$$

$$= \frac{1}{2} - \frac{1}{4}x + \frac{1}{8}x^2 - \frac{1}{16}x^3.$$

2. Construction of the Taylor series of an unknown function

Construct the degree-4 Taylor polynomial approximation (at 0) of the solution of the initial value problem

$$y'(t) = \left(y(t)\right)^2, \qquad y(0) = -3$$

Do this by repeatedly differentiating the differential equation to obtain y', y'', y''', and $y^{(4)}$. You'll need to use the chain and product rules of differentiation.

$$y(0) = -3$$

$$y' = y^2 \qquad y'(0) = (-3)^2 = 9$$

$$y'' = 2yy' \qquad y''(0) = 2\cdot(-3)(9) = -54$$

$$y''' = 2y'y' + 2yy''$$

$$= 2(y')^2 + 2yy'' \qquad y'''(0) = 2(9)^2 + 2(-3)(-54)$$

$$= 162 + 324$$

$$= 486$$

$$y^{(4)} = 4y'y'' + 2y'y'' + 2yy'''$$

$$= 6y'y'' + 2yy''' \qquad y^{(4)}(0) = 6(9)(-54) + 2(-3)(486)$$

$$= -5832$$

$$y(t) \approx -3 + 9t - \frac{54}{2}t^2 + \frac{486}{3!}t^3 - \frac{5832t^4}{4!}$$

$$= -3 + 9t - 27t^2 + 81t^3 - 243t^4$$