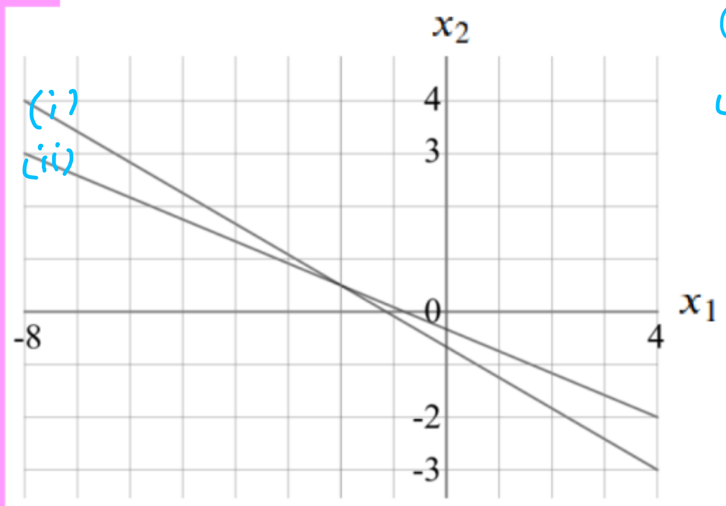
Exam 2 Fall 2025 Solutions

1 The equations of the two lines in the sketch below are



(i) $x_2 = 4 - \frac{7}{12}(x_1 + 8)$

(ii) $x_2 = 3 - \frac{5}{12}(x_1 + 8)$

(a) Write this system in matrix vector form $Ax = b$.

(b) Compute $||A||_\infty$ and $||A^{-1}||_\infty$

(c) Compute the infinity-norm condition number of matrix A. Based on this number, is the linear system well conditioned or not? Explain.

(d) Citing an appropriate theorem, compute a bound on the norm of the change in the solution if δA is zero and $||\delta b||/||b|| \le 0.01$.

(a)

$$\begin{bmatrix} \frac{7}{12} & 1 \\ \frac{5}{12} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 - \frac{56}{12} \\ 3 - \frac{40}{12} \end{bmatrix}$$

(b) $\|A\|_\infty = \max_i \sum_j |a_{ij}| = \frac{7}{12} + 1 = \frac{19}{12}$

$A^{-1} = \frac{1}{\frac{7}{12} - \frac{5}{12}} \begin{bmatrix} 1 & -1 \\ -\frac{5}{12} & \frac{7}{12} \end{bmatrix} = \frac{12}{2} \begin{bmatrix} 1 & -1 \\ -\frac{5}{12} & \frac{7}{12} \end{bmatrix} = \begin{bmatrix} 6 & -6 \\ -\frac{5}{2} & \frac{7}{2} \end{bmatrix}$. So $\|A^{-1}\|_\infty = 6 + 6 = 12$

(c) $\kappa_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty = \frac{19}{12} \cdot 12 = 19.$

Roughly speaking errors or changes in the data A, b will be amplified by as much as a factor of 19.
This is not ideal (the ideal is no amplification at all : $\kappa = 1$) but not terrible. (See part (d).)

(d) We have Thm 8.10: $\frac{\|\delta x\|}{\|x\|} \le \frac{\kappa(A)}{(1 - \|\delta A\| \|A^{-1}\|)} \left(\frac{\|\delta b\|}{\|b\|} + \frac{\|\delta A\|}{\|A\|} \right)$

If $\delta A = 0$, $\frac{\|\delta x\|}{\|x\|} \le \frac{\kappa(A)}{1} \frac{\|\delta b\|}{\|b\|}$

If $\kappa(A) = 38$ and $\frac{\|\delta b\|}{\|b\|} \le 0.01$, then $\frac{\|\delta x\|}{\|x\|} \le 19 \cdot 0.01 = .19$

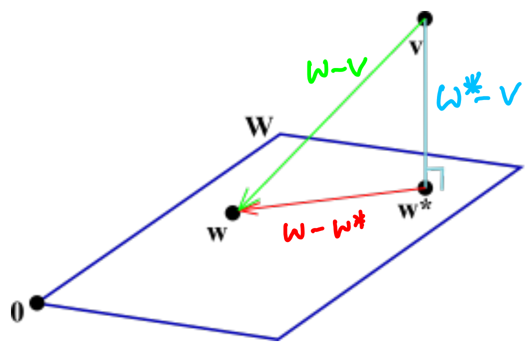
Thus a 1% change in b can yield no more than a 19% change in x.

2 Let V be a vector space with an innner product $(\cdot, \cdot) \to \mathbb{R}^+$, and let $||\cdot||$ be the norm induced by this inner product. That is, $||v||^2 = (v, v)$.

Let W be a subspace of V, and let $v \in V$, $w^* \in W$ be such that, with respect to the inner product above, $w^* - v$ is orthogonal to every $w \in W$.

Show that no vector $w \in W$ is closer to v than w^* is.

That is, show that $||v - w|| \geq ||v - w^*|| \; \forall w \in W$.



Want to show that

$$\|w-v\| \geq \|w^*-v\| \quad \forall w \in W.$$

$$w-v = w^*-v + w-w^*$$

So

$$\|w-v\|^2 = (w-v, w-v)$$

$$= (w^*-v + w-w^*, \; w^*-v + w-w^*)$$

$$= (w^*-v, w^*-v) + \underbrace{\cancel{(w^*-v, w-w^*)}}_{0} + \underbrace{\cancel{(w-w^*, w^*-v)}}_{0} + (w-w^*, w-w^*)$$

because $w-w^* \in W$ and $w^*-v \in W_\perp$ by hypothesis

$$= \|w^*-v\|^2 + \|w-w^*\|^2$$

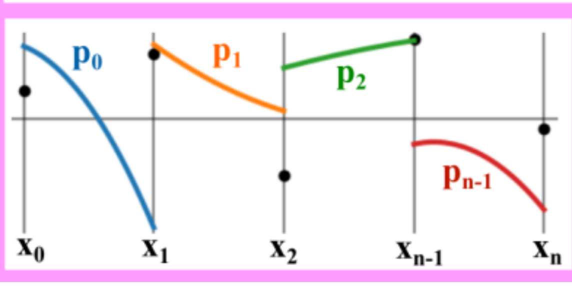
$$\geq \|w^*-v\|^2$$ because this term is non-negative. QED.

3

Recall that the cubic spline interpolant is a piecewise cubic function whose value and first and second derivatives are continuous at the nodes (and everywhere else). With n subintervals, these requirements impose $4n-2$ constraints on the $4n$ parameters of the n cubic pieces, which are supplemented by 2 additional constraints, such as the "natural" or "clamped" end conditions.

(a) If we chose instead to build a piecewise-**quadratic** interpolant to data $\{(x_j, y_j)\}_{j=0,1,\dots,n}$ with nodes $a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$, write down an appropriate set of constraints, and compare the total number of constraints with the number of parameters.

(b) Can you suggest a sensible way of supplementing the constraints to complete the specification of the interpolant? If so, do so. If not, say why not.



(a) We have n quadratic pieces $p_i(x) = a_i + b_i x + c_i x^2$, $i=1,\dots,n$.
That's a total of $\boxed{3n \text{ parameters}}$ (degrees of freedom).
For interpolation, we need

$$\left.\begin{array}{ll} p_i(x_i) = y_i, & i=0,\dots,n-1 \to n \text{ constraints} \\ p_i(x_{i+1}) = y_{i+1}, & i=0,\dots,n-1 \to n \text{ constraints} \end{array}\right\} 2n \text{ constraints so far}$$

We can accommodate n more constraints.
We could make the interpolant C^1 by requiring derivative matching at the interior nodes:

$$p_{i-1}'(x_i) = p_i'(x_i), \quad i=1,\dots,n-1 \to (n-1) \text{ constraints.}$$

We now have $\boxed{3n-1}$ equations in $3n$ variables:
we need 1 more equation!

(b) Can't impose any condition at both ends.
I don't have any bright ideas off the top of my head for a natural additional condition. Maybe that's why we don't hear about quadratic splines?

Devise a Householder reflector (an orthogonal matrix) H_1 which will zero-out the subdiagonal portion of the first column of the matrix A below. State the projector P that you use in constructing H_1, and give the product $H_1 A$.

$$\begin{bmatrix} 3 & 5 \\ 0 & 1 \\ 4 & 7 \end{bmatrix}$$

No part of the calculation generates numbers with more than one digit after the decimal point.

$$z = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}, \quad \|z\|_2 = \sqrt{3^2+4^2} = 5, \quad w = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} \text{ (or its negative)}$$

$$v \equiv w - z = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -4 \end{bmatrix}.$$

$$v^T v = (2)^2 + (-4)^2 = 20. \quad v v^T = \begin{bmatrix} 2 \\ 0 \\ -4 \end{bmatrix} \begin{bmatrix} 2 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 4 & 0 & -8 \\ 0 & 0 & 0 \\ -8 & 0 & 16 \end{bmatrix}$$

$$P = \frac{v v^T}{v^T v} = \begin{bmatrix} .2 & 0 & -.4 \\ 0 & 0 & 0 \\ -.4 & 0 & .8 \end{bmatrix}$$

$$H_1 = I - 2P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} .4 & 0 & -.8 \\ 0 & 0 & 0 \\ -.8 & 0 & 1.6 \end{bmatrix} = \boxed{\begin{bmatrix} .6 & 0 & .8 \\ 0 & 1 & 0 \\ .8 & 0 & -.6 \end{bmatrix}}$$

$$H_1 A = \begin{bmatrix} .6 & 0 & .8 \\ 0 & 1 & 0 \\ .8 & 0 & -.6 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 0 & 1 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 5 & 3.0+5.6 \\ 0 & 1 \\ 0 & 4.0-4.2 \end{bmatrix} = \begin{bmatrix} 5 & 8.6 \\ 0 & 1 \\ 0 & -0.2 \end{bmatrix}$$

no calculation required

With respect to the inner product $(f, g) = \int_0^1 f(x)g(x)dx$, find the least squares approximation to $f(x) = e^x$ on [0,1] within the subspace of $C[0, 1]$ spanned by the functions $\{1, x\}$.

Our approximation is of the form $p(x) = c_0 \cdot 1 + c_1 \cdot x$.

We require the error $p-f$ to be orthogonal to each of the basis functions.

Thus (i) $\int_0^1 (c_0 + c_1 x - e^x) \cdot 1 \, dx = 0$

and (ii) $\int_0^1 (c_0 + c_1 x - e^x) x \, dx = 0$

(i) $\left| c_0 x + c_1 \frac{x^2}{2} - e^x \right|_0^1 = 0$

$\left(c_0 + \frac{c_1}{2} - e\right) - (0 + 0 - 1) = 0 \longrightarrow \boxed{c_0 + \frac{c_1}{2} = e - 1}$

(ii) $\int_0^1 x e^x dx \overset{\text{by parts}}{=} x e^x \Big|_0^1 - \int_0^1 e^x dx = e - e^x \Big|_0^1 = e - (e-1) = 1$

So $\left| c_0 \frac{x^2}{2} + c_1 \frac{x^3}{3} \right|_0^1 - 1 = 0 \longrightarrow \frac{1}{2} c_0 + \frac{1}{3} c_1 = 1$

$\boxed{c_0 + \frac{2}{3} c_1 = 2}$

Subtracting (i) from (ii):

$\underbrace{\left(\frac{2}{3} - \frac{1}{2}\right)}_{\frac{1}{6}} c_1 = 3 - e \rightarrow c_1 = 18 - 6e$

Then $c_0 = 2 - \frac{2}{3}(18 - 6e) = 2 - (12 - 4e) = -10 + 4e$

And so $\boxed{p(x) = -10 + 4e + (18 - 6e)x}$