1. Optimization in multiple dimensions

Consider the problem of finding a local minimizer of the function $f(x, y) = -x^3 + y^3 + 3x^2 - 3y^2 + 12$.

(a) By hand, and showing all your steps, calculate the formula for f along the initial 1D line search - in either steepest descent or conjugate gradient methods - starting at (x,y) = (-1,1), and find the local minimizer closest to your starting point.

(b) For the staring point (-1,1), by hand sketch on the following contour plot of f (i) the first 3 steepest-descent line minimizations, (ii) the curve resulting from following the local gradient at every point. Recall that the gradient is orthogonal to the level curves.

$$\nabla f(x,y) = (-3x^2 + 6x, 3y^2 - 6y)$$

$$\nabla f(-1,1) = (-3 \cdot 1 + 6(-1), 3-6) = (-9,-3)$$

$$-\nabla f(-1,1) = (9,3)$$

1st line minimization is along

$$\ell(t) = (-1,1) + (9,3)t = (-1+9t, 1+3t)$$

or equivalently along $(x_0,y_0) - \frac{1}{3}\nabla f(x_0,y_0)t$

$$\ell(t) = (-1,1) + (3,1)t = (-1+3t, 1+t).$$

$$f(\ell(t)) = -(-1+3t)^3 + (1+t)^3 + 3(-1+3t)^2 - 3(1+t)^2 + 12$$

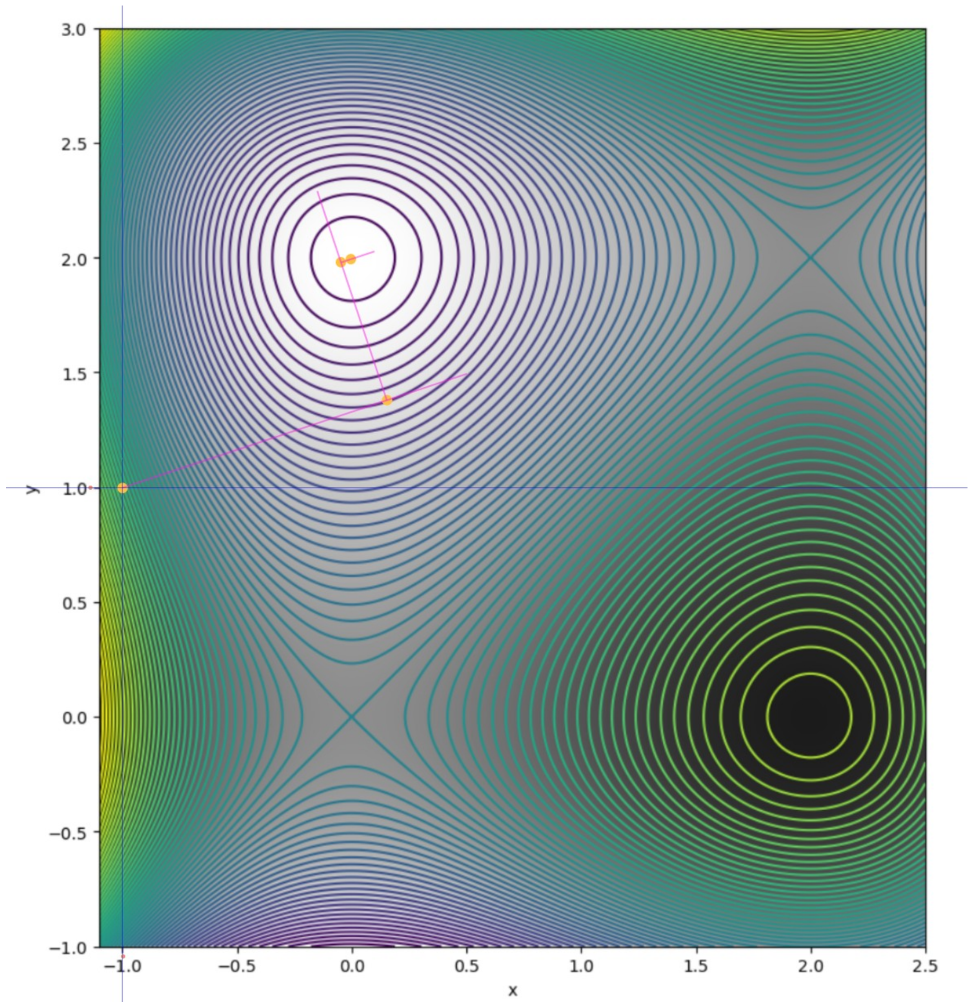
$$= -26t^3 + 54t^2 - 30t + 14$$

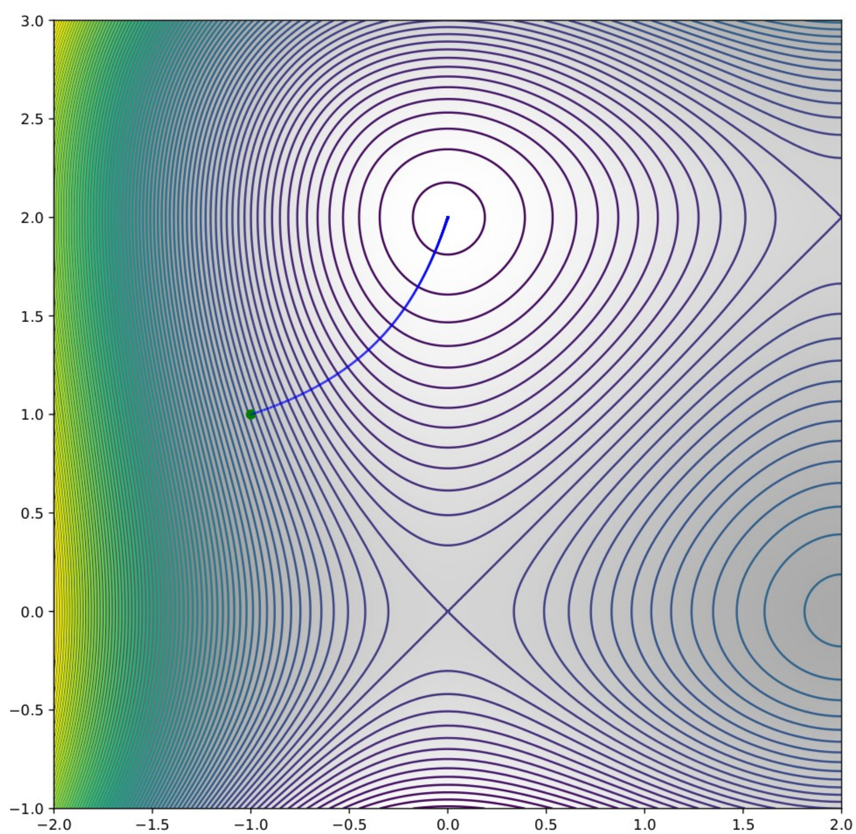
$$\frac{d}{dt} f(\ell(t)) = -78t^2 + 108t - 30 \overset{\text{set}}{=} 0$$

$$\xrightarrow{\text{quad formula}} t = \frac{5}{13} \text{ or } 1, \text{ with } \frac{5}{13} \text{ closer.}$$

$$\ell\left(\frac{5}{13}\right) = \boxed{\left(\frac{2}{13}, \frac{18}{13}\right) =: x^{(1)}} \text{ minimizes } f \text{ along search line}$$

(b) I actually computed these curves. Hand-drawn ones should agree approximately.





(c) To apply Newton's method to grad f, we will need the jacobian of grad f, a.k.a. the hessian of f:

$$\nabla f(x,y) = (-3x^2+6x,\ 3y^2-6y)$$

$$D\nabla f(x,y) = \begin{bmatrix} -6x+6, & 0 \\ 0, & 6y-6 \end{bmatrix}$$

At $(x^{(0)}, y^{(0)}) = (-1, 1)$:

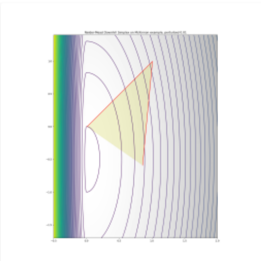
$$D\nabla f(-1,1) = \begin{bmatrix} 6+6, & 0 \\ 0 & 6-6 \end{bmatrix} = \begin{bmatrix} 12 & 0 \\ 0 & 0 \end{bmatrix}.$$

We observe that at $(-1,1)$, the hessian is singular, so the Newton step from here is undefined. Newton fails.

This is a caution that while Newton usually converges very rapidly when it does converge, it is not as robust as other methods.

Nelder-Mead failure on McKinnon's example.

2. I re-ran the code I showed in class with several perturbed initial simplices. The method converged successfully in all instances. Here is one example: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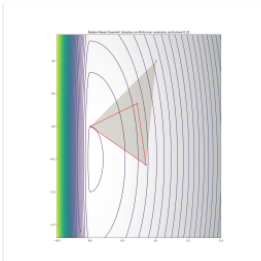
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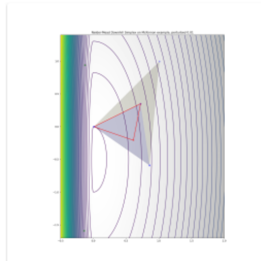
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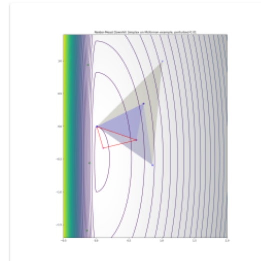
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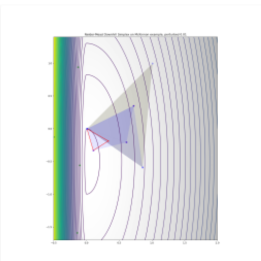
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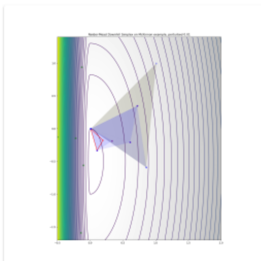
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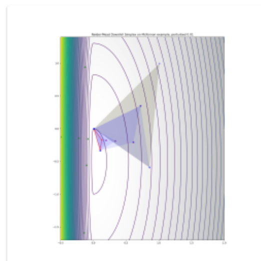
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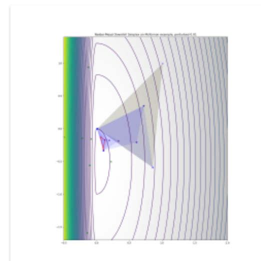
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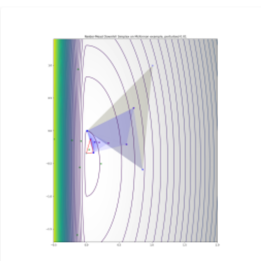
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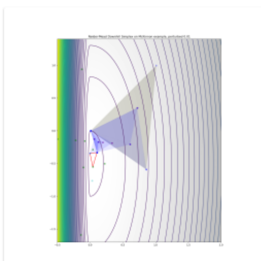
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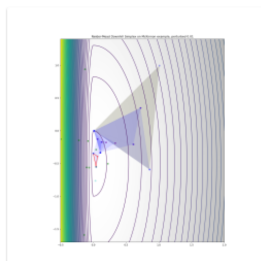
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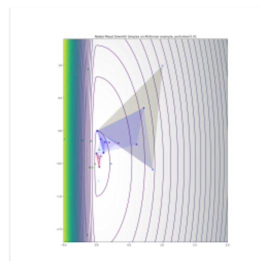
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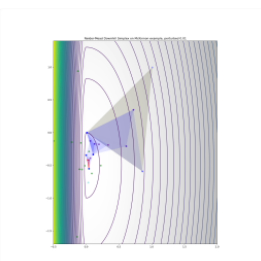
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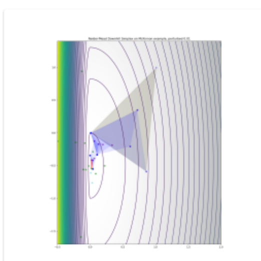
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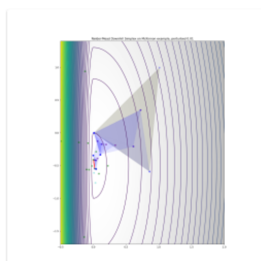
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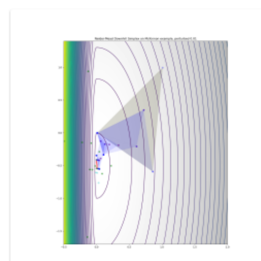
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These results suggest that it is only for special functions and starting simplices that the method fails in this manner.